

Courant Institute of
Mathematical Sciences
Magneto-Fluid Dynamics Division

RF Energy Absorption Due
to the Continuous Spectrum
of Ideal Magnetohydrodynamics

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Abstract

The relationship between energy absorption and the continuous frequency spectrum of the linearized equations of ideal magnetohydrodynamics is investigated. We limit our considerations to incompressible fluid perturbations for which the continuum stems from resonant surfaces where the oscillation frequency equals the local frequency of an Alfvén wave. Details of this absorption process are illustrated by obtaining the response

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p. 32: Lines 12 - 14 should be replaced by the following sentence:
"It was pointed out in the previous section that when the applied force varies sinusoidally in time, the fields in the plasma also vary sinusoidally in time with constant amplitude except at the point $x = x_0(\omega_0)$ where the fields grow in time."

p. 36: In line 13, the number "0.07" should be replaced by "7."

After submission of the report for printing, the following reference concerning Alfvén wave heating was referred to the author:

Hasegawa, A., Chen, L. 1974. Phys. Rev. Letters, 32, 454.

Abstract

The relationship between energy absorption and the continuous frequency spectrum of the linearized equations of ideal magnetohydrodynamics is investigated. We limit our considerations to incompressible fluid perturbations for which the continuum stems from resonant surfaces where the oscillation frequency equals the local frequency of an Alfvén wave. Details of this absorption process are illustrated by obtaining the response of the diffuse sheet pinch to an external current source. Expressions are derived which give the rate at which energy is transferred to the plasma and show the spatial distribution of the absorbed energy. The results indicate that the absorption is enhanced if the plasma profile has regions of large spatial gradients. This enhanced absorption is a consequence of the presence of surface waves.

1. Introduction

Conservative, dynamical systems under the linear approximation have the property that they can display dissipative effects if there exists a sufficiently large number of degrees of freedom (Sturrock 1961; Grad 1969). In simplest terms apparent dissipation arises when the system has a continuous frequency spectrum. The normal modes which make up the continuum can phase-mix in time, leading to decay of macroscopic variables. Plasma models, free of any apparent loss mechanism, are often encountered which are characterized by a continuum and are consequently dissipative. Among these models we mention a thermal plasma described by the linearized Vlasov equation (Landau 1952; Weitzner 1963), a cold non-uniform plasma (Barston 1964) and certain configurations, such as the screw pinch (Grad 1973; Goedbloed and Sakanaka 1974), the theta pinch (Weitzner 1973; Grossmann and Tataronis 1973) and the sheet pinch (Boris 1968; Tataronis and Grossmann 1972; Uberoi 1972; Tataronis and Grossmann 1973), described by the linearized equations of ideal magnetohydrodynamics. (In what follows the initials MHD will designate the term magnetohydrodynamics).

A second implication of a continuum is that the system can irreversibly absorb energy when under the influence of an externally applied force. The absorption is a consequence of a resonance which occurs when the spatial distribution and the time behavior of the force matches a mode in the continuum. A characteristic of this absorption is that it occurs locally

in the space in which the continuum is defined. This relationship between a continuous spectrum and energy absorption has been recognized in the past (Grad 1969; Baldwin and Ignat 1969; Tataronis and Grossmann 1972; Hasegawa 1973), and calculations have been made to determine the effectiveness of this method of energy transfer as a means of heating plasmas (Baldwin and Ignat 1969; Brownell 1972; Tataronis 1973; Grossmann, Kaufmann and Neuhauser 1973).

In what follows we shall be concerned with this absorption mechanism under the approximation of ideal MHD.[†] We shall limit our considerations to a particular plasma configuration, namely the diffuse sheet pinch, for which it has already been shown that the linearized MHD equations have a continuous spectrum (Boris 1968; Tataronis and Grossmann 1972; Uberoi 1972; Tataronis and Grossmann 1973). Under the approximation that the fluid motion is incompressible, we obtain the response of the plasma to an external sheet current located in a surrounding vacuum region and determine from the response the energy transferred to the plasma. The force which drives the plasma is derived from the pressure exerted on the plasma surface by the magnetic field induced in the vacuum by the sheet current and the plasma current. As we shall demonstrate, this scheme leads to efficient coupling of energy to the interior of the plasma region even though the plasma motion is

[†] The results of this paper were first reported at the October 1973 meeting of the Plasma Physics Division of the American Physical Society, Philadelphia, Pa. (Abstract in Bull. Am. Phys. Soc. 18, 1291 (1973)).

incompressible. The absorption rate is measured in terms of the effective impedance that the plasma imposes on the sheet current. It will be shown that as a consequence of the continuum, the impedance has a resistive component which, for parameters and dimensions typical of high beta, high density plasmas, can have values exceeding one Ohm at frequencies on the order of 2 MHz. This resistance is comparable to what has been found for excitation of the lower hybrid resonance by a method similar to the one assumed here (Puri 1973), with the advantage that the absorption described below occurs at a relatively low frequency.

2. The Plasma Model and Basic Equations

The specific plasma model we study is the planar sheet pinch with the equilibrium magnetic field $B_z (\equiv B_0)$ parallel to the z axis of an (x,y,z) Cartesian coordinate system as illustrated in Fig. 1. The equilibrium pressure $p(x)$ and mass density $\rho(x)$, as well as $B_z(x)$, depend only on x and are symmetric with respect to $x = 0$. The plasma is confined to the region $-b < x < b$. For $|x| > b$, there is a vacuum where $\rho \equiv 0$, B_z is independent of x and an externally supported sheet current J_s is present which is directed parallel to the y axis and depends sinusoidally on z ,

$$J_s(x,y,z,t) = J_s(t)[\delta(x-c) - \delta(x+c)]\cos kz \hat{y} \quad (1)$$

In this expression k is a real wave number, $\delta(x)$ is the Dirac delta function J_s has the dimensions of current per unit area and $J_s(t)$ is expressed in terms of current per unit length, length being measured in the z direction. The sheet current is the source of the external force which acts on the plasma surface.

We describe the plasma dynamics by means of the linearized equations of ideal MHD with resistivity and viscosity neglected and incompressibility assumed. In Maxwell's equations, the displacement current is neglected. Since the assumed equilibrium magnetic field is free of curvature, we have $(B_z \cdot \nabla)B_z = 0$, implying that the MHD equations can be reduced to the following form:

$$\rho \frac{\partial^2 \chi}{\partial t^2} - \frac{1}{\mu} (\mathbf{B} \cdot \nabla)^2 \chi = -\nabla \frac{\partial p}{\partial t}^* \quad (2)$$

$$\nabla \cdot \chi = 0 \quad (3)$$

$$\mathbf{E} = -\chi \times \mathbf{B} \quad (4)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E} \quad (5)$$

$$\mathbf{J}_1 = \frac{1}{\mu} \nabla \times \mathbf{B}_1 \quad (6)$$

where μ is the vacuum permeability, χ , $\mathbf{B}_1$, and $\mathbf{E}$ are perturbations of the velocity, magnetic and electric fields, and p^* represents a perturbation of the total pressure, i.e. $p^* = p_1 + \mu^{-1} \mathbf{B} \cdot \mathbf{B}_1$. In the vacuum the fluctuating electric, $\delta \mathbf{E}_V$, and magnetic, $\delta \mathbf{B}_V$, fields are determined by,

$$\delta \mathbf{B}_V = \nabla \phi \quad (7)$$

$$\frac{\partial \delta \mathbf{B}_V}{\partial t} = -\nabla \times \delta \mathbf{E}_V \quad (8)$$

where the magnetic potential ϕ satisfies,

$$\nabla^2 \phi = 0. \quad (9)$$

At $x = b$, the plasma and vacuum fields are connected by the usual continuity conditions of ideal MHD, namely $\delta \mathbf{B}_V$ is parallel to the fluctuating plasma-vacuum interface and the total pressure $p + B^2/\mu$ is continuous across the interface. After

linearization, these two conditions imply the following equalities at $x = b$,

$$\mathbf{n} \times \delta \mathbf{E}_V = (\mathbf{n} \cdot \mathbf{v}) \mathbf{B}_V \quad (10)$$

$$\frac{1}{\mu} \mathbf{B}_V \cdot \delta \mathbf{B}_V = p^* \quad (11)$$

where $\mathbf{B}_V$ ($\equiv B_V \hat{z}$) is the equilibrium magnetic field in the vacuum region and $\mathbf{n}$ is a unit vector normal to the plasma-vacuum interface of the equilibrium and directed into the vacuum. Finally, the following jump conditions across the sheet current are to be satisfied,

$$\mathbf{n} \times [\delta \mathbf{B}_V] = \mu \mathbf{J}_S \quad (12)$$

$$\mathbf{n} \times [\delta \mathbf{E}_V] = 0 \quad (13)$$

where the brackets designate the jump in the enclosed quantity.

The symmetry of the sheet current implies that the excited mode is characterized as follows,

$$\mathbf{v} = (v_x, 0, v_z), \quad (14)$$

$$\mathbf{E} = (0, E_y, 0), \quad (15)$$

$$\mathbf{B}_1 = (B_x, 0, B_z), \quad (16)$$

$$\mathbf{J}_1 = (0, J_y, 0), \quad (17)$$

$$\delta \mathcal{B}_V = (\delta B_x, 0, \delta B_z), \quad (18)$$

$$\delta \mathcal{E}_V = (0, \delta E_y, 0) \quad (19)$$

where in each expression the parenthesis on the right enclose the x , y , and z Cartesian components of the vector on the left. Furthermore, it can be readily established that the induced fields possess a spatial symmetry with respect to the plane $x = 0$, namely,

$$v_x(-x, z, t) = -v_x(x, z, t), \quad v_z(-x, z, t) = v_z(x, z, t), \quad (20)$$

$$E_y(-x, z, t) = -E_y(x, z, t), \quad (21)$$

$$B_x(-x, z, t) = -B_x(x, z, t), \quad B_z(-x, z, t) = B_z(x, z, t), \quad (22)$$

$$J_y(-x, z, t) = -J_y(x, z, t), \quad (23)$$

$$p^*(-x, z, t) = p^*(x, z, t). \quad (24)$$

$\delta \mathcal{B}_V$ and $\delta \mathcal{E}_V$ have the same symmetry as $\mathcal{B}_1$ and $\mathcal{E}$ respectively. Finally, since the imposed surface current, eq. (1), depends on z in the form $\cos kz$, it follows that v_x , B_z , E_y , J_y , p^* , δB_z and δE_y also have the form $f(x, t) \cos kz$, while v_z , B_x and δB_x depend on z as $g(x, t) \sin kz$. In what follows the dependence of the fields on z as either $\cos kz$ or $\sin kz$ will not be explicitly shown except in cases where ambiguity would result by leaving the z dependence implicit.

In the sheet pinch geometry, the plasma fields and energy can be expressed explicitly in terms of v_x . In particular, by combining eqs. (2) and (3), one finds,

$$v_z = -\frac{1}{k} \frac{\partial v_x}{\partial x} (x,t) \sin kz, \quad (25)$$

$$\frac{\partial p^*}{\partial t} = -\frac{1}{k^2} [\rho(x) \frac{\partial^2}{\partial t^2} + \frac{1}{\mu} k^2 B^2(x)] \frac{\partial v_x}{\partial x} (x,t) \cos kz, \quad (26)$$

where v_x satisfies,

$$\frac{\partial}{\partial x} \left[\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\mu} k^2 B^2 \right) \frac{\partial v_x}{\partial x} \right] - k^2 \left(\rho \frac{\partial^2}{\partial t^2} + \frac{1}{\mu} k^2 B^2 \right) v_x = 0. \quad (27)$$

The remaining fields are obtained from v_x and v_z with the aide of eqs. (4)-(6). As a consequence of the spatial symmetry indicated by eqs. (20)-(24), it suffices to solve eq. (27) in the region $x > 0$ subject to boundary conditions at $x = 0$,

$$v_x(0, z, t) = 0, \quad (28)$$

and at $x = b$,

$$v_x(b, z, t) = \frac{1}{B_V} \delta E_y(b, z, t), \quad (29)$$

$$p^*(b, z, t) = \frac{1}{\mu} B_V \delta B_z(b, z, t), \quad (30)$$

where $\delta E_y(x, z, t)$ and $\delta B_z(x, z, t)$ satisfy eqs. (7)-(9), are consistent with eq. (12) and (13), and vanish as $x \rightarrow \infty$.

The equation which expresses the rate of change of the plasma energy in terms of v_x is derived by introducing $\xi(x, z, t)$, the displacement of a fluid element from its equilibrium position,

$$\xi = \int_0^t dt' \chi \quad (31)$$

By integrating eq. (2) with respect to t , with the initial conditions set equal to zero, scalar multiplying by χ and the integrating over some plasma volume V bounded by a surface Σ , one finds the following statement of energy conservation in terms of ξ and χ ,

$$\int_V d\mathcal{V} \left[\frac{1}{2} \rho \frac{\partial \chi^2}{\partial t} - \frac{1}{\mu} \chi \cdot (\mathcal{B} \cdot \nabla)^2 \xi \right] = - \oint_{\Sigma} d\mathcal{A} \cdot \chi p^* \quad (32)$$

where use was made of the incompressibility condition $\nabla \cdot \chi = 0$. For the volume V , we choose the region bounded on the x axis by $x = \pm b$, on the y axis by $y = 0$ and $y = L_y$, where L_y is an arbitrarily chosen dimension, and on the z axis by $z = 0$ and $z = 2\pi/k$. By taking into consideration that v_x and p^* are antisymmetric in x and depend on z in the form $\cos kz$ while v_z is symmetric in x and depends on z as $\sin kz$, the surface integration and the integration over y and z in the left hand side of eq. (32) can be readily accomplished, yielding,

$$\frac{dW}{dt} = -\frac{2\pi L_y}{k} v_x(b,t) p^*(b,t), \quad (33)$$

where W , the plasma energy, is given by,

$$W = \frac{2\pi L_y}{k} \int_0^b dx \left[\frac{1}{2} \rho(x) \chi^2(x,t) + \frac{1}{2\mu} k^2 B^2(x) \xi^2(x,t) \right]. \quad (34)$$

χ^2 signifies $v_x^2 + v_z^2$, with a similar definition for ξ^2 .

Equation (33) states that the rate at which the plasma energy increases in the enclosed volume is equal to the energy flux density, $v_x(b,t)p^*(b,t)$, averaged over y and z at $x = \pm b$, multiplied by the surface area $2\pi L_y/k$. It follows from the boundary conditions given by eqs. (29) and (30) that the energy flux density is equal to the x component of the vacuum Poynting vector $S_x \equiv \left[\delta E_v \times \delta B_v / \mu \right]_x$ at the plasma boundary, which in turn can be related by means of Maxwell's equations to the rate at which the energy of the perturbed vacuum magnetic field W_v changes in time and to the power delivered by the sheet current, $-(2\pi L_y/k) \delta E_y(c,t) J_s(t)$. Explicitly, eq. (33) is equivalent to the following expressions,

$$\frac{dW}{dt} = - \frac{2\pi L_y}{k} \frac{\delta E_y(b,t) \delta B_z(b,t)}{\mu} \quad (35)$$

$$= - \frac{dW_v}{dt} - \frac{2\pi L_y}{k} \delta E_y(c,t) J_s(t), \quad (36)$$

where,

$$W_v = \frac{2\pi L_y}{k} \int_b^\infty dx \frac{\delta B_v^2}{2\mu}(x,t) \quad (37)$$

Equations (25)-(30) and (33)-(37) form the basis of the remaining sections of the paper. In section 3 we solve eq. (27) for v_x and use this solution to obtain expressions for the plasma and vacuum fields in terms of the surface current. Section 4 is devoted to an examination of the temporal behavior and spatial distribution of the energy absorbed by the plasma, and finally in section 5 we examine the absorption rate for a specific plasma profile.

3. Solution of MHD Equations

In this section we make use of the Laplace transform, defined by,

$$v_x(x, t) = \int_{\mathcal{L}} \frac{d\omega}{2\pi} e^{i\omega t} \hat{v}_x(x, \omega) , \quad \text{Im}(\omega) < 0 , \quad (38)$$

$$\hat{v}_x(x, \omega) = \int_0^{\infty} dt e^{-i\omega t} v_x(x, t) , \quad (39)$$

to solve the linearized MHD equations for the plasma and vacuum fields. In eq. (38), $\mathcal{L}$ is a contour of integration below the singularities of the integrand in the lower half complex ω plane. The differential equation which $\hat{v}_x$ satisfies is obtained by transforming eq. (27) with eq. (39), yielding,

$$\frac{\partial}{\partial x} [A(x, \omega) \frac{\partial \hat{v}_x}{\partial x}] - k^2 A(x, \omega) \hat{v}_x = 0 , \quad \text{Im}(\omega) < 0 , \quad (40)$$

where,

$$A(x, \omega) = -\rho(x)\omega^2 + k^2 B^2(x)/\mu , \quad (41)$$

and initial conditions have been set equal to zero. From eqs. (25) and (26), the Laplace transforms of $v_z(x, t)$ and $p^*(x, t)$ are,

$$\hat{v}_z(x, \omega) = -\frac{1}{k} \frac{\partial \hat{v}_x}{\partial x}(x, \omega) , \quad (42)$$

$$\hat{p}^*(x, \omega) = -\frac{A(x, \omega)}{i\omega k^2} \frac{\partial \hat{v}_x}{\partial x}(x, \omega) . \quad (43)$$

The vacuum fields, defined by eqs. (7)-(9), can be explicitly expressed in terms of the exponential function $e^{\pm kx}$. In particular, the vacuum electric field which satisfies the jump conditions at the sheet current, and vanishes as $x \rightarrow \infty$, is, after performing a Laplace transform,

$$\hat{\delta E}_y(x, \omega) = \begin{cases} -i\omega \frac{\mu \hat{J}_s(\omega)}{k} e^{-kc} \sinh kx + D(\omega)e^{-kx} , & b < x < c \quad (44) \\ -i\omega \frac{\mu \hat{J}_s(\omega)}{k} e^{-kx} \sinh kc + D(\omega)e^{-kx} , & x > c \quad (45) \end{cases}$$

where $D(\omega)$ is a function of ω determined by the boundary conditions at $x = b$. The electric field has been written as the sum of two terms: the first term represents the electric field induced in the vacuum directly by the sheet current, while the second term is the component of the electric field resulting from the plasma motion. If the plasma is removed, then the second term vanishes. The boundary conditions at $x = b$, eqs. (10) and (11), can be expressed in terms of $\hat{v}_x$ and $\hat{E}_y$. After a Laplace transformation is performed, they read,

$$\hat{v}_x(b, \omega) = \frac{1}{B_v} \delta \hat{E}_y(b, \omega) , \quad (46)$$

$$\frac{\partial v_x}{\partial x}(b, \omega) = \frac{k^2 B_v}{\mu A(b, \omega)} \frac{\partial \delta \hat{E}_y}{\partial x}(b, \omega) . \quad (47)$$

At $x = 0$ we have,

$$\hat{v}_x(0, \omega) = 0 . \quad (48)$$

Equations (40)-(48) form a self-consistent system which determines $\hat{v}_x$ and $\hat{E}_y$.

To solve the system we introduce a function $\psi(x, \omega)$ which satisfies eq. (40), is free of singularities on the real x axis in the range $0 \leq x \leq b$ for ω on the Laplace contour, and satisfies the following conditions at $x = 0$,

$$\psi(0, \omega) = 0, \quad (49)$$

$$\frac{\partial \psi}{\partial x}(0, \omega) = k. \quad (50)$$

$\hat{v}_x(x, \omega)$ must then be proportional to ψ ,

$$\hat{v}_x(x, \omega) = C(\omega)\psi(x, \omega) \quad (51)$$

The constants $C(\omega)$ and $D(\omega)$ are obtained by substituting eqs. (44) and (51) into eqs. (46) and (47) yielding for $\text{Im}(\omega) < 0$,

$$C(\omega) = -i\omega \frac{\mu \hat{J}_s(\omega)}{B_v} \frac{k^2 B_v^2}{\mu A(b, \omega)} \frac{e^{-k(c-b)}}{[1 + kq(\omega)]\psi'(b, \omega)} \quad (52)$$

$$D(\omega) = i\omega \frac{\mu \hat{J}_s(\omega)}{k} \frac{\sinh kb - kq(\omega) \cosh kb}{1 + kq(\omega)} e^{-k(c-b)}, \quad (53)$$

where $\psi'(x, \omega)$ signifies $d\psi(x, \omega)/dx$ and,

$$q(\omega) \equiv \frac{k^2 B_v^2}{\mu A(b, \omega)} \frac{\psi(b, \omega)}{\psi'(b, \omega)}. \quad (54)$$

It should be noted that q and $\hat{v}_x$ are independent of the normalization imposed on ψ .

ψ can be expressed in terms of known functions only in certain cases. One example is the case where A is independent of x , for which eq. (40) has the solution,

$$\psi = \sinh kx, \quad (55)$$

A second example is the piecewise smooth profile shown in Fig. 2. In this case the plasma equilibrium is uniform in the region $|x| < a$, while in the sheath $a < |x| < b$, $\rho(x)$ and $B^2(x)$ vary linearly with x , implying,

$$A(x, \omega) = \begin{cases} -\omega^2 \rho_p + \frac{k^2 B_p^2}{\mu}, & -a \leq x \leq a, \\ -\omega^2 \rho_p \frac{b-|x|}{b-a} + \frac{k^2}{\mu} [B_v^2 + (B_v^2 - B_p^2) \frac{|x|-b}{b-a}], & a \leq |x| \leq b \end{cases} \quad (56)$$

and,

$$\psi(x, \omega) = \begin{cases} \sinh kx, & 0 < x < a, \\ \sinh ka \frac{p(\omega) I_0[k(x_0-x)] - K_0[k(x_0-x)]}{p(\omega) I_0[k(x_0-a)] - K_0[k(x_0-a)]}, & a < x < b \end{cases} \quad (57)$$

where $I_0(z)$ and $K_0(z)$ are modified Bessel functions of order zero as defined by Watson (Watson 1958), $x_0(\omega)$ is defined by $A(x_0, \omega) = 0$, and,

$$p(\omega) = \frac{K_0[k(x_0-a)] - K_1[k(x_0-a)] \tanh ka}{I_0[k(x_0-a)] + I_1[k(x_0-a)] \tanh ka}. \quad (58)$$

At $x = a$, we imposed the constraint that ψ/ψ' be continuous.

More generally, if $A(x, \omega)$ possesses a power series expansion in some domain about $x_0(\omega)$ which includes $0 \leq x \leq b$,

$$A(x, \omega) = \sum_{n=1}^{\infty} a_n (x - x_0)^n, \quad (59)$$

then for $a_1 \neq 0$, i.e. for $A'(x_0, \omega) \neq 0$, it can be shown that about x_0 , two independent solutions of eq. (40) are,

$$f_1 = g(x, \omega), \quad (60)$$

$$f_2 = g(x, \omega) \ln(x_0 - x) + (x - x_0)h(x, \omega), \quad (61)$$

with which we form,

$$\psi = p(\omega)g(x, \omega) + [g(x, \omega) \ln(x_0 - x) + (x - x_0)h(x, \omega)], \quad (62)$$

$p(\omega)$ is independent of x , and $g(x, \omega)$ and $h(x, \omega)$ are functions analytic in some domain about $x_0(\omega)$, the extent of the domain being determined by the closest zero of $A(x, \omega)$ to x_0 . By analytic continuation, this domain can be extended to include $x = 0$ where the boundary conditions (49) and (50) are imposed. This uniquely determines g , h and p .

Both eqs. (57) and (62) show that when A depends on x , ψ possesses a logarithmic singularity at the point where $x_0(\omega) = x$. Since the origin and significance of this singularity are considered elsewhere (Tataronis and Grossmann 1973; Grossmann and Tataronis 1973), we shall limit our comments here concerning it by mentioning that the singularity can be related to the

existence of a continuous spectrum in the frequency range where $A(x, \omega)$ vanishes at some point $x_0(\omega)$ in the plasma region. From eq. (41), the continuum has the following bounds,

$$\min[\omega_a^2(x)] < \omega^2 < \max[\omega_a^2(x)] , \quad (63)$$

where $\omega_a^2(x) \equiv k^2 B^2(x) / \mu \rho(x)$ is the square of the local frequency of an Alfvén wave propagating with wave number k . If ω satisfies eq. (63), then there exists a point $x_0(\omega)$ in the plasma, defined by

$$\omega^2 = \omega_a^2(x_0) , \quad (64)$$

about which ψ and consequently $\hat{v}_x$ possess a logarithmic singularity. By differentiating $\hat{v}_x$ with respect to x to form $\hat{v}_z$ according to eq. (42), one finds that $\hat{v}_z$ is also singular about x_0 ,

$$\hat{v}_z \sim \frac{1}{x - x_0(\omega)} , \quad (65)$$

but the singularity is stronger than that of $\hat{v}_x$ in the sense that when ω is in the continuum, $\hat{v}_z$ is not square integrable with respect to x . It is the fact that $\hat{v}_z$ is not square integrable which implies the absorption process that is the subject of the present paper.

The presence of a logarithmic factor renders ψ a multi-valued function. Branch cuts must then be introduced in the complex ω plane such that with ω on the principal branch,

ψ is free of singularities on the real x axis in the range $-b < x < b$. For the piecewise smooth profile in Fig. 2, ψ is given by eq. (57), and the associated branch points and branch cuts are shown in Fig. 3. On the positive real axis of the ω plane, ψ is real for $\omega > \omega_a(x)$. Due to the logarithmic factor in K_0 , a branch point is present at $\omega = \omega_a(x)$, with the corresponding branch cut defined by,

$$\omega - \omega_a(x) = |\omega - \omega_a(x)|e^{i\theta}, \quad \pi \geq \theta \geq -\pi. \quad (66)$$

It can be shown that for $x_0 > a$, eq. (66) implies,

$$x_0(\omega) - x = |x_0(\omega) - x|e^{i\theta}, \quad \pi \geq \theta \geq -\pi, \quad (67a)$$

$$\psi = \sinh ka \frac{[p(\omega)+i\epsilon\pi]I_0[k(x_0-x)]-K_0[k|x_0-x|]}{p(\omega)I_0[k(x_0-a)]-K_0[k(x_0-a)]} \quad (67b)$$

where ω is real, and $\epsilon = +1, -1$ or 0 for $\theta = +\pi, -\pi$ or 0 . Therefore, along the branch cut, ψ is complex with the sign of its imaginary part dependent on whether ω lies on the upper ($\theta=\pi$) or lower ($\theta=-\pi$) edge of the cut. The branch cut terminates at the second branch point $\omega = \omega_p$. For $0 < \omega < \omega_p$, ψ is again real. The behavior of ψ on the negative ω axis is given by the symmetry relation,

$$\psi(x, -\omega) = \psi^\dagger(x, \omega), \quad (68)$$

where $\psi^\dagger$ designates the complex conjugate of ψ . As we will show in the next section, the imaginary part of ψ plays an

important role in demonstrating that the plasma absorbs energy when the applied frequency is in the continuum.

If $\omega_a^2(x)$ is an analytic function, one can construct the branch cuts of ψ by using either eq. (62) or more conveniently the following representation of ψ ,

$$\psi = Q(\omega)[p(\omega)\psi_1(x,\omega) + \psi_2(x,\omega)], \quad (69)$$

where Q and p are functions of ω determined by the boundary conditions at $x = 0$, and ψ_1 and ψ_2 are two linearly independent solutions of eq. (40) which we write in the following form,

$$\psi_1 = g(x,\omega), \quad (70)$$

$$\psi_2 = g(x,\omega) \int_a^x \frac{dx'}{A(x',\omega)g^2(x',\omega)}.$$

$g(x,\omega)$ is analytic about x_0 and is assumed to have the normalization $g(x_0,\omega) = 1$. For a particular value of x , $x = x_p$, the branch cuts of $\psi(x_p,\omega)$ coincide with those ranges of ω in which $A(x,\omega)$ vanishes somewhere for $0 < x < x_p$. The branch points are located at $\omega = \pm\omega_a(x_p)$ and at those values of $\omega_a(x)$ at which $d\omega_a^2/dx = 0$. To establish the existence of a branch point at $\omega_a(x_p)$, we expand $\omega_a^2(x)$ about x_p ,

$$\omega_a^2(x) = \omega_a^2(x_p) + \frac{d\omega_a^2}{dx}(x_p)(x-x_p) + \dots \quad (72)$$

to derive the following expansion of $x_0(\omega)$ about $\omega_a(x_p)$,

$$x_o(\omega) \approx x_p + \frac{\omega_a^2(x_p)}{(d\omega_a^2/dx)_{x=x_p}} + \dots \quad (73)$$

Assuming $(d\omega_a^2/dx)_{x=x_p} > 0$, eq. (73), together with,

$$\omega - \omega_a(x_p) = |\omega - \omega_a(x_p)|e^{i\theta}, \quad \pi \geq \theta \geq -\pi \quad (74)$$

implies the following behavior of $\psi_2(x_p, \omega)$ on the real ω axis near $\omega_a(x_p)$,

$$\psi_2(x_p, \omega) = g(x_p, \omega) P \int_a^{x_p} \frac{dx'}{A(x', \omega) g^2(x', \omega)} + \frac{i\epsilon \pi g(x_p, \omega)}{\rho(x_p) (d\omega_a^2/dx)_{x=x_p}}, \quad (75)$$

where P designates that the principal value of the integral is to be taken when x_o lies on the real x' axis, and,

$$\epsilon = \begin{cases} 1, & \text{for } \theta = \pi \\ 0, & \text{for } \theta = 0 \\ -1, & \text{for } \theta = -\pi \end{cases} \quad (76)$$

Equation (75) implies the existence of a branch cut on the real axis for $\omega > \omega_a(x_p)$. Since the boundary conditions which ψ must satisfy at $x = 0$ are real, it is readily established with eq. (75) that $\psi(x_p, \omega)$ is real for $\omega > \omega_a(x_p)$ while complex along the branch cut. A similar conclusion follows for $(d\omega_a^2/dx)_{x_p}$.

Let us now assume that $d\omega_a^2/dx = 0$ at $x = x_v$ and that about x_v , $\omega_a^2(x)$ can be expanded in the following manner,

$$\omega_a^2(x) = \omega_v^2 + \beta_v(x-x_v)^2 + \dots \quad (77)$$

Equation (77) implies that for $\omega \approx \omega_v$, $A(x, \omega)$ possesses two zeros, $x_{v1}(\omega)$ and $x_{v2}(\omega)$, which have the following expansions,

$$\begin{aligned} x_{v1}(\omega) &\approx x_v + \left(\frac{\omega^2 - \omega_v^2}{\beta_v}\right)^{1/2} + \dots \\ x_{v2}(\omega) &\approx x_v - \left(\frac{\omega^2 - \omega_v^2}{\beta_v}\right)^{1/2} + \dots \end{aligned} \quad (78)$$

Assuming $\beta_v > 0$, corresponding to a local minimum of $\omega_a^2(x)$, eq. (78) combined with,

$$\omega - \omega_v = |\omega - \omega_v| e^{i(\theta - 2\pi)}, \quad 0 \leq \theta \leq 2\pi \quad (79)$$

implies for $x_v < a < x_p$ and ω real,

$$\psi_2(x_p, \omega) = g(x_p, \omega)^P \int_a^{x_p} \frac{dx'}{A(x', \omega) g^2(x', \omega)} + \frac{i\epsilon\pi}{2} \frac{g(x_p)}{\rho(x_p)\beta_v} \left| \left(\frac{\beta_v}{\omega^2 - \omega_v^2} \right)^{1/2} \right| \quad (80)$$

where,

$$\epsilon = \begin{cases} -1, & \text{for } \theta = 2\pi \\ 0, & \text{for } \theta = \pi \\ 1, & \text{for } \theta = 0 \end{cases} \quad (81)$$

A similar expression can be derived for ψ_2 if $\beta_v < 0$, corresponding to a local maximum of $\omega_a^2(x)$. The implication of eq. (80) is that ψ possesses a branch point at ω_v , with the branch cut extending to the region $\omega > \omega_v$. Figure 4 indicates the branch cuts which are implied by the above considerations when

$\omega_a^2(x)$ depends on x in the manner shown in Fig. 4a. It can be shown that when $\omega_a^2(x)$ is analytic, eq. (68) is satisfied for real ω .

We now use the explicit forms of ψ derived above to carry out the inverse Laplace transform of $\hat{v}_x(x, \omega)$ to obtain $v_x(x, t)$. In eq. (38) the integration on the Laplace contour $\mathcal{L}$ can be performed by exploiting the singularities of the integrand in the complex ω plane. The procedure we follow is standard in the theory of Laplace transforms, namely, we deform the contour $\mathcal{L}$ from its position in the lower half ω plane below the singularities of $\hat{v}_x(x, t)$ to a contour $\mathcal{L}'$ which encircles the singularities and extends into the upper half plane. Even if the integration cannot be explicitly carried out with the aid of this new contour, one can obtain with $\mathcal{L}'$ a time asymptotic expansion of the integral by expanding the integrand about the singularities and then performing the integration for large t . We follow this procedure for the case where the sheet current varies sinusoidally in time,

$$J_s(t) = J_0 \sin \omega_0 t, \quad \hat{J}_s(\omega) = J_0 \frac{\omega_0}{\omega_0^2 - \omega^2}. \quad (82)$$

After combining eqs. (38), (51), (52) and (82) and deforming $\mathcal{L}$ to $\mathcal{L}'$ in the upper half plane, one obtains for $\omega_0^2 \neq \omega_a^2(x)$,

$$v_x(x, t) = J_0 R_v(x, \omega_0) \sin \omega_0 t + J_0 X_v(x, \omega_0) \cos \omega_0 t + v_{xt}(x, t). \quad (83)$$

R_v and X_v are real functions defined by,

$$R_V + 1 X_V \equiv Z_V, \quad (84)$$

where,

$$Z_V(x, \omega) \equiv \frac{\hat{v}_x}{J_s} = - \frac{1\omega\mu}{B_V} \frac{k^2 B_V^2}{\mu A(b, \omega)} \frac{e^{-k(c-b)}}{1 + kq(\omega)} \frac{\psi(x, \omega)}{\psi'(b, \omega)}, \quad (85)$$

and v_{xt} is the transient term,

$$v_{xt}(x, t) = \int_{\mathcal{L}'} \frac{d\omega}{2\pi} e^{i\omega t} Z_V(x, \omega) \hat{J}_s(\omega). \quad (86)$$

As illustrated in Fig. 5, $\mathcal{L}'$ extends into the upper half plane around the singularities of Z_V . Z_V is characterized by two sets of singularities. The first set consists of the branch points and corresponding branch cuts of ψ on the real axis. In Fig. 5 the branch cuts have been analytically continued into the upper half plane. By performing this analytic continuation, one obtains the second set of singularities, namely, poles at ω_s where,

$$\Delta(\omega_s) \equiv 1 + kq(\omega_s) = 0. \quad (87)$$

It is pointed out that under the incompressible approximation Δ does not possess zeros on the principal sheet of the ω plane. Zeros are found only after an analytic continuation has been performed across the logarithmic branch cut. The significance of the roots of eq. (87) are discussed in considerable detail elsewhere (Tataronis and Grossmann 1973; Grossmann and Tataronis 1973). Therefore, we shall limit our comments here concerning them by indicating that ω_s is in general complex and corresponds

to the frequency of a surface wave which decays exponentially in time due to phase mixing. This decay is analogous to the decay of space charge waves in a Vlasov plasma by the mechanism of "Landau damping" .

As $t \rightarrow \infty$, one finds

$$v_{xt}(x,t) \sim \alpha(x) \frac{\cos \omega_a(x)t}{\omega_a(x)t} + \sum_v \gamma_v(x) \frac{\cos \omega_{av}t}{(\omega_{av}t)^{n_v}} + S(x)e^{-\omega_{si}t} \cos \omega_{sr}t \quad (88)$$

where ω_{sr} and ω_{si} are the real and imaginary parts of ω_s , and α , γ_v and S are functions of x obtained by expanding $Z_v(x, \omega)$ about each singularity and then performing the integration over ω as $t \rightarrow \infty$. Although the surface wave decays exponentially in time, it can dominate the transient term in some finite time interval. The parameter which determines the extent of this time interval is the effective length L of the region in which the plasma equilibrium is inhomogeneous. For the profile shown in Fig. 2, $L = (b - a)$. As $L \rightarrow 0$, the plasma profile approaches one having a sharp boundary. In this limit one can show that $R_v \equiv 0$ in the forced component of v_x while in the transient, α , γ_v and ω_{si} are identically zero. Therefore if there is a sharp boundary, the transient term consists of an undamped surface wave and is of the same order as the forced term. This behavior can be explained in terms of the condensation of the continuous spectrum of the inhomogeneous plasma into a point spectrum as $L \rightarrow 0$.

The transient term of the other fields which characterize the system can exhibit a time behavior which differs from that of v_{xt} . For example, by differentiating eq. (88) to form v_{zt} , the transient of v_z , according to eq. (25), one finds that as $t \rightarrow \infty$,

$$v_{zt} \sim \beta(x) \frac{d\omega_a}{dx}(x) \sin \omega_a(x)t, \quad (89)$$

i.e., v_{zt} oscillates in time without attenuation at the local Alfvén frequency. The transient of the current density grows linearly in time,

$$J_{yt} \sim \delta(x) \left[\frac{d\omega_a}{dx}(x) \right]^2 t \cos \omega_a(x)t. \quad (90)$$

The expressions that are given above are not valid at the point x_0 where $\omega_0^2 = \omega_a^2(x_0)$. The reason for this is that the singularities of Z_v and $\hat{J}_s$ superimpose to form a singularity of higher order in $\hat{y}_x$. Thus, at $x = x_0(\omega_0)$,

$$\hat{v}_x(x_0, \omega) \equiv Z_v(x_0, \omega) \hat{J}_s(\omega) \sim \frac{\ln(\omega - \omega_0)}{\omega - \omega_0}, \quad (91)$$

which implies,

$$v_x(x, t) \sim \ln t \cos \omega_0 t, \quad (92)$$

as $t \rightarrow \infty$. For the same reason, it be shown that at x_0 ,

$$v_z \sim t \cos \omega_0 t, \quad (93)$$

$$J_y \sim t^2 \cos \omega_0 t, \quad (94)$$

The temporal growth which occurs at x_0 will be associated in the next section with local energy absorption at that point.

4. Energy Absorption

In this section we shall be concerned with the behavior of the plasma energy W in the limit $t \rightarrow \infty$. In particular we wish to determine the conditions under which dW/dt has a non-zero average value when the surface current varies sinusoidally in time. If the average of dW/dt over the period of the applied frequency is positive, then one can conclude that the plasma absorbs energy.

Equation (33) expresses dW/dt in terms of v_x and p^* at the plasma-vacuum boundary. In the limit $t \rightarrow \infty$, $v_x(b, t)$ is obtained from eq. (83),

$$v_x(x, t) = J_0 R_v(b, \omega_0) \sin \omega_0 t + J_0 X_v(b, \omega_0) \cos \omega_0 t, \quad (95)$$

where

$$R_v(b, \omega_0) + iX_v(b, \omega_0) = -\frac{i\omega_0 \mu}{kB_v} e^{-k(c-b)} + \frac{i\omega_0 \mu}{kB_v} \frac{e^{-k(c-b)}}{1+kq(\omega_0)}, \quad (96)$$

while $p^*(b, t)$ can be derived by taking the inverse Laplace transform of eq. (43),

$$p^*(b, t) = J_0 R_p(b, \omega_0) \sin \omega_0 t + J_0 X_p(b, \omega_0) \cos \omega_0 t \quad (97)$$

where,

$$R_p(b, \omega_0) + iX_p(b, \omega_0) = B_v \frac{e^{-k(c-b)}}{1+kq(\omega_0)}. \quad (98)$$

By combining eqs. (33), (95) and (97), the following expression for the time rate of change of the plasma energy is obtained,

$$\begin{aligned} \frac{dW}{dt} = & - \frac{2\pi L_y}{k} \frac{J_0^2}{2} \frac{\omega_0 \mu q_1(\omega_0)}{|1+kq(\omega_0)|^2} e^{-2k(c-b)} \\ & + \frac{2\pi L_y}{k} \frac{J_0^2}{2} [R_v(b, \omega_0) R_p(b, \omega_0) - X_v(b, \omega_0) X_p(b, \omega_0)] \cos 2\omega_0 t \\ & - \frac{2\pi L_y}{k} \frac{J_0^2}{2} [R_p(b, \omega_0) X_v(b, \omega_0) + R_v(b, \omega_0) X_p(b, \omega_0)] \sin 2\omega_0 t, \end{aligned} \quad (99)$$

where $q_1(\omega_0)$ is the imaginary part of $q(\omega_0)$. According to eq. (99), dW/dt is given by the sum of a term which is constant in time and terms which oscillate sinusoidally in time at twice the applied frequency. The constant term is the average of dW/dt over the period $T_0 \equiv 2\pi/\omega_0$,

$$\left\langle \frac{dW}{dt} \right\rangle = - \frac{2\pi L_y}{k} \frac{\omega_0 \mu q_1(\omega_0)}{|1+kq(\omega_0)|^2} \frac{J_0^2}{2} e^{-2k(c-b)}, \quad (100)$$

and the time varying terms are due to the energy of the plasma fields which have saturated in magnitude. A necessary condition which must be satisfied in order that $q_1(\omega_0)$ be different from zero is that ω_0 lies in the continuum, i.e., there must exist a real point $x_0(\omega_0)$ in the plasma at which $\omega_0^2 = \omega_p^2(x_0)$. If this condition is not satisfied, then the initial conditions $\psi(0, \omega_0) = 0$ and $\psi'(0, \omega_0) = k$ imply ψ is real for $x > 0$ which in turn implies that q is real. If ω_0 is in the continuum, then $q_1(\omega_0)$ can be expressed in terms of the Wronskian of the independent solutions, f_1 and f_2 , of eq. (40), where about $x_0(\omega_0)$, f_1 and f_2 are given by eqs. (60) and (61), and the

Wronskian is,

$$\begin{aligned}\hat{W}(f_1, f_2) &\equiv f_1 \frac{df_2}{dx} - \frac{df_1}{dx} f_2 \\ &= \frac{\rho(x_0) g^2(x_0, \omega_0)}{A(x, \omega_0)} \frac{d\omega_a^2}{dx}(x_0) \quad .\end{aligned}\quad (101)$$

Let us assume that $\omega_a^2(x)$ is a monotonic function of x for $0 \leq x \leq b$, so that there exists only one point at which $\omega_0^2 = \omega_a^2(x)$. Then the dependence of $q_1(\omega_0)$ on $\hat{W}$ is established by analytically continuing eq. (62) to the region $x > x_0(\omega_0)$ according to the following prescription,

$$x_0(\omega_0) - x = [x - x_0(\omega_0)]e^{-i\epsilon\pi} \quad , \quad (102)$$

$$\psi(x, \omega_0) = [p(\omega_0) - i\epsilon\pi]g(x, \omega_0)$$

$$+ [\ln |x_0(\omega_0) - x| g(x, \omega_0) + (x - x_0(\omega_0))h(x, \omega_0)] ,$$

where $\epsilon = \text{sgn}(d\omega_a^2/dx)_{x_0(\omega_0)}$, and constructing $\psi(x, \omega_0)/\psi'(x, \omega_0)$. This yields,

$$\text{Im} \left(\frac{\psi}{\psi'} \right) = -\epsilon\pi \frac{\hat{W}(x, \omega_0)}{|\psi'(x, \omega_0)|^2} \quad , \quad (103)$$

where use has been made of eq. (102) with $\theta = -\pi$. Substitution of this expression in eq. (54) yields,

$$q_1(\omega_0) = -\pi \frac{k^2 B_v^2}{\mu A^2(b, \omega_0)} \frac{\rho(x_0) g^2(x_0, \omega_0)}{|\psi'(b, \omega_0)|^2} \left| \frac{d\omega_a^2}{dx}(x_0) \right| \quad , \quad (104)$$

where use was made of the identity $\epsilon (d\omega_a^2/dx) \equiv \left| d\omega_a^2/dx \right|$.

Since $q_1 < 0$, eq. (100) implies that $\langle dW/dt \rangle > 0$ and consequently that the plasma is an absorptive medium in a frequency range which coincides with the continuous spectrum.

If there exists more than one real $x_0(\omega_0)$ at which $\omega_0^2 = \omega_a^2(x)$, the analytic continuation defined by eq. (102) can be performed at each such point $x_0^v(x_0)$, $v = 1, 2, \dots$, to obtain q_1 . Specifically, about x_0^v , we have,

$$\psi \equiv \psi_v \equiv Q_v(\omega_0)[p_v(\omega_0)f_{1v}(x, \omega_0) + f_{2v}(x, \omega_0)] , \quad (105)$$

where,

$$f_{1v} = g_v(x, \omega_0) , \quad (106)$$

$$f_{2v} = g_v(x, \omega_0) \exp(x_0^v - x) + (x - x_0^v)h_v(x, \omega_0) . \quad (107)$$

g_v and h_v are analytic at x_0^v and normalized such that they are real for ω_0 real. Let a_v be a point which is less than x_v but in the domain where eq. (105) is valid. Then, p_v is determined by imposing continuity of ψ/ψ' at $x = a_v$, yielding

$$p_v(\omega) = - \frac{f_{2v}(a_v, \omega) - \alpha_{v-1} f'_{2v}(a_v, \omega)}{f_{1v}(a_v, \omega) - \alpha_{v-1} f'_{1v}(a_v, \omega)}$$

where,

$$\alpha_{v-1} \equiv \frac{\psi_{v-1}(a_v, \omega)}{\psi'_{v-1}(a_v, \omega)} . \quad (109)$$

To obtain Q_v , we impose continuity of ψ at $x = a_v$, but since $q(\omega)$ is independent of Q_v , its form will not be of concern to us here. Analogous to what appears in eq. (102), the analytic continuation of ψ_v from $x < x_0^v$ to $x > x_0^v$ proceeds as follows,

(110)

$$\begin{aligned}\psi_v &= Q_v(\omega)[p_v(\omega)g_v(x, \omega) + g_v(x, \omega)\ln(x_0^v - x) + (x - x_0^v)h_v(x, \omega)], \\ &\quad x < x_0^v \\ &= Q_v(\omega)\{[p_v(\omega) - i\epsilon_v\pi]g_v(x, \omega) + g_v(x, \omega)\ln(x - x_0^v) + (x - x_0^v)h_v(x, \omega)\}, \\ &\quad x > x_0^v \\ &\equiv Q_v(\omega)\{[p_v(\omega) - i\epsilon_v\pi]f_{1v}^*(x, \omega) + f_{2v}^*(x, \omega)\}, \quad x > x_0^v\end{aligned}$$

where $\epsilon_v \equiv \text{sgn} (d\omega_a^2/dx)_{x=x_0^v(\omega)}$, and $\tilde{f}_{1v}$ and $\tilde{f}_{2v}$ are analytic continuations of $g_v(x, \omega)$ and $g_v(x, \omega)\ln(x - x_0^v) + (x - x_0^v)h_v(x, \omega)$ respectively in the region $x > x_0^v$. Let p_{v1} be the imaginary part of p_v . Then by making use of eq. (110), it can be shown that,

$$\text{Im} \left[\frac{\psi_v(x, \omega)}{\psi_v^*(x, \omega)} \right] = [p_{v1}(\omega) - \epsilon_v\pi] \frac{\tilde{W}[f_{1v}^*(x, \omega), f_{2v}^*(x, \omega)]}{|[p_v(\omega) - i\epsilon_v\pi]f_{1v}^*(x, \omega) + f_{2v}^*(x, \omega)|^2}, \quad (111)$$

where from eq. (108),

$$p_{v1}(\omega) = \alpha_{(v-1)i} \frac{\tilde{W}[f_{1v}(a_v, \omega), f_{2v}(a_v, \omega)]}{|f_{1v}(a_v, \omega) - \alpha_{v-1}f_{1v}'(a_v, \omega)|^2}. \quad (112)$$

$\alpha_{(v-1)i}$ is the imaginary part of $\alpha_{(v-1)}$. The Wronskians of f_{1v} , f_{2v} and f'_{1v} , f'_{2v} have the same functional dependence on x , i.e.,

$$\left. \begin{aligned} \tilde{W}(f_{1v}, f_{2v}) \\ \tilde{W}(f'_{1v}, f'_{2v}) \end{aligned} \right\} = \frac{\rho(x_o^v) g_v^2(x_o^v, \omega)}{A(x, \omega)} \frac{d\omega_a^2}{dx}(x_o^v). \quad (113)$$

By combining eqs. (109), (111), (112) and (113), the following recursion relation for α_{vi} is found,

$$\alpha_{vi} = -\epsilon_v \pi \phi_v \rho(x_o^v) \left| \frac{d\omega_a^2}{dx}(x_o^v) \right| - \alpha_{(v-1)i} \Theta_v \phi_v \rho^2(x_o^v) \left| \frac{d\omega_a^2}{dx}(x_o^v) \right|^2, \quad (114)$$

where,

$$\Theta_v \equiv \frac{g_v^2(x_o^v, \omega)}{|A| |f_{1v} - \alpha_{v-1} f'_{1v}|^2} \Big|_{x=a_v}, \quad \phi_v \equiv \frac{g_v^2(x_o^v, \omega)}{|A| |(p_v - i\epsilon_v \pi) f'_{1v} + f'_{2v}|^2} \Big|_{x=a_{v+1}}. \quad (115)$$

Let $x_o^n(\omega_o)$ be the zero of A which is the closest to the plasma-vacuum interface at $x = b$. Then, $q_1(\omega_o)$ is obtained by substituting eq. (111) into eq. (54) with v set equal to n , yielding,

$$\begin{aligned} q_1(\omega_o) = & -\pi \frac{k^2 B_v^2}{\mu A^2(b, \omega_o)} \frac{\rho(x_o^n) g_n^2(x_o^n, \omega_o) |Q_n(\omega_o)|^2}{|\psi'_n(b, \omega_o)|^2} \left| \frac{d\omega_a^2}{dx}(x_o^n) \right| \\ & - \epsilon_n \alpha_{(n-1)i} \frac{k^2 B_v^2}{\mu A^2(b, \omega_o)} \frac{\rho^2(x_o^n) g_n^2(x_o^n, \omega_o) |Q_n(\omega_o)|^2}{|\psi'_n(b, \omega_o)|^2} \Theta_n \left| \frac{d\omega_a^2}{dx}(x_o^n) \right|^2. \end{aligned} \quad (116)$$

$\alpha_{(n-1)\pm}$ follows from the recursion relation eq. (114) with the initial condition $\alpha_{0i} = 0$. Eq. (114) can also be used to show that each singular point $x_0^v(\omega_0)$ adds a negative definite term to eq. (116). Therefore, $q_i(\omega_0)$ is negative definite which, according to eq. (100), implies $\langle dW/dt \rangle > 0$.

Asymptotically in time, the energy absorbed by the plasma is transferred to an infinitesimally narrow region about $x_0(\omega_0)$ and is partitioned so as to increase predominately the plasma motion along the z axis in this region. To show this we shall obtain below $\langle dW/dt \rangle$ by carrying out the integration over x in eq. (34).

It was pointed out in the previous section that when the applied force varies sinusoidally with constant amplitude as $t \rightarrow \infty$ except at $x = x_0(\omega_0)$ where the fields grow in time. Therefore, the only region which can give rise to a non-zero average of dW/dt is some narrow region about x_0 . Moreover since the singular eigenfunctions of v_x are square integrable over x , the contribution to $W(t)$ from v_x and ξ_x approaches a constant. Therefore, to leading order as $t \rightarrow \infty$, we obtain with eq. (34),

$$\left\langle \frac{dW}{dt} \right\rangle \approx \lim_{\Delta x \rightarrow 0} \left\langle \frac{d}{dt} \int_{x_0(\omega_0) - \Delta x}^{x_0(\omega_0) + \Delta x} dx \left[\frac{1}{2} \rho(x) v_z^2(x, t) + \frac{1}{2\mu} k^2 B^2(x) \xi_z^2(x, t) \right] \right\rangle \frac{2\pi L_y}{k} \quad (117)$$

v_z and ξ_z are found by combining eqs. (42), (51) and (52) and then performing an inverse Laplace transform. For $x \approx x_0(\omega_0)$, this leads to, in the limit $t \rightarrow \infty$,

$$\begin{aligned}
v_z &\approx -|\tilde{C}(\omega_0)|g(x_0, \omega_0) \frac{d\omega_a^2}{dx}(x_0) \frac{J_0}{k\omega_0} \frac{\sin \frac{1}{2}[\omega_a(x) - \omega_0]t}{\omega_a(x) - \omega_0} \sin \omega_0 t \\
\xi_z &\approx |\tilde{C}(\omega_0)|g(x_0, \omega_0) \frac{d\omega_a^2}{dx}(x_0) \frac{J_0}{k\omega_0^2} \frac{\sin \frac{1}{2}[\omega_a(x) - \omega_0]t}{\omega_a(x) - \omega_0} \cos \omega_0 t
\end{aligned} \tag{118}$$

where $\tilde{C}$ is related to C by,

$$C(\omega) = \tilde{C}(\omega) \hat{J}_0(\omega) \tag{119}$$

Combining eqs. (117) and (118) and integrating over x yields for large t ,

$$\left\langle \frac{dW}{dt} \right\rangle \approx \frac{2\pi L_y}{k} \frac{\pi |\tilde{C}(\omega_0)|^2 g^2(x_0, \omega_0)}{k^2 \omega_0} \rho(x_0) \left| \frac{d\omega_a^2}{dx}(x_0) \right| \frac{J_0^2}{2} . \tag{120}$$

It can be shown that eq. (120) is in agreement with the expression obtained by combining eqs. (100) and (104).

5. Coil Impedance

In an experimental situation where the plasma resonance is excited by means of the current in a coil surrounding the plasma, a measurable quantity which determines both the power absorbed by the plasma and the energy stored outside of the resonant region is the coil impedance $Z(\omega)$. Since the current in the coil is effectively a surface current, the impedance is given by the ratio $-\delta\hat{E}_y(\omega)/\hat{J}_s(\omega)$, where $\hat{J}_s$ and $\delta\hat{E}$ are respectively the Laplace transforms of the surface current and the vacuum electric field parallel to $\hat{J}_s$. For the planar model in Fig. 1, we have,

$$Z(\omega) \equiv - \frac{\delta\hat{E}_y(c, \omega)}{\hat{J}_s(\omega)} \quad (121)$$

$$= i \frac{\omega\mu}{k} e^{-kc} \sinh kc - i \frac{\omega\mu}{k} \frac{\sinh kb - kq(\omega) \cosh kb}{1 + kq(\omega)} e^{-k(2c-b)}$$

where $\delta\hat{E}_y(c, \omega)$ is expressed in terms of $\hat{J}_s(\omega)$ by eqs. (44) and (53). When $J_s(t)$ varies sinusoidally in time in the manner shown by eq. (82), the average power absorbed by the plasma as $t \rightarrow \infty$ is related to Z as follows,

$$P = \frac{2\pi L_y}{k} \operatorname{Re}[Z(\omega_0)] \frac{J_0^2}{2}, \quad (122)$$

where from eq. (121),

$$\operatorname{Re}[Z(\omega)] = - \frac{\omega\mu q_1(\omega)}{|1 + kq(\omega)|^2} e^{-2k(c-b)} \quad (123)$$

Equations (100) and (122) are seen to be in agreement. The imaginary part of Z determines the total average energy in the

plasma and vacuum regions outside the resonance zones. To illustrate in what manner Z typically depends on ω , let us consider the piece-wise smooth profile in Fig. 2. For this case, q has an explicit representation in terms of Bessel functions that can be obtained by combining eq. (54) and (67), where θ is to be set equal to $-\pi$ in eq. (67). This yields,

$$kq(\omega) = \frac{[p(\omega)-i\pi]I_0[k(b-x_0)]-K_0[k(b-x_0)]}{[p(\omega)-i\pi]I_1[k(b-x_0)]+K_1[k(b-x_0)]} \quad , \quad (124)$$

where $p(\omega)$ is given by eq. (58) and,

$$x_0 - a = \frac{\omega^2 \rho_p - k^2 B_p^2 / \mu}{\omega^2 \rho_p + k^2 (B_v^2 - B_p^2) / \mu} (b-a), \quad b-x_0 = \frac{k^2 B_v^2 / \mu}{\omega^2 \rho_p + k^2 (B_v^2 - B_p^2) / \mu} (b-a) \quad (125)$$

It can be shown with eqs. (58) and (124) that q is complex in the continuum $\omega > \omega_p (\equiv k^2 B_p^2 / \rho_p \mu)$, but real for $0 < \omega < \omega_p$. The imaginary part of q which is to be substituted into eq. (123) is given by,

$$kq_1(\omega) = - \frac{\pi}{k(b-x_0)} \frac{1}{|[p(\omega)-i\pi]I_1[k(b-x_0)]+K_1[k(b-x_0)]|^2} \quad . \quad (126)$$

The real and imaginary parts of Z are plotted in Fig. 6 as a function of $f(\equiv \omega/2\pi)$ for a hydrogen and a deuterium plasma. The parameter values chosen are typical of a high beta, high density plasma. The real part of Z , Z_r , is identically zero up to the low frequency end of the continuum at $f = f_p$. For $f > f_p$, Z_r peaks at about 2 MHz for a deuterium plasma and at $2 \cdot 8$ MHz for a hydrogen plasma, and then

approaches zero as $f \rightarrow \infty$. In the vicinity of the peak, the form of $Z(\omega)$ suggests the existence of a damped resonance. This resonant behavior is due to the presence of a complex zero of the denominator, $1 + kq(\omega)$, of eq. (121) near the real ω axis. This zero is the complex frequency of the damped surface wave which was discussed in section 3.

Fig. 6 shows that the resistive component of the coil impedance has a peak value of about one Ohm. To estimate the efficiency of the absorption process, one should compare this value with the surface resistance of the coil. For example, the surface resistance of copper wire at 2 MHz is about 0.0007 Ohm, implying that if the coil is made of copper, only 0.07 per cent of the power delivered to the coil would be lost to Joule heating of the wire, while 93 per cent of the power would be absorbed by the plasma.

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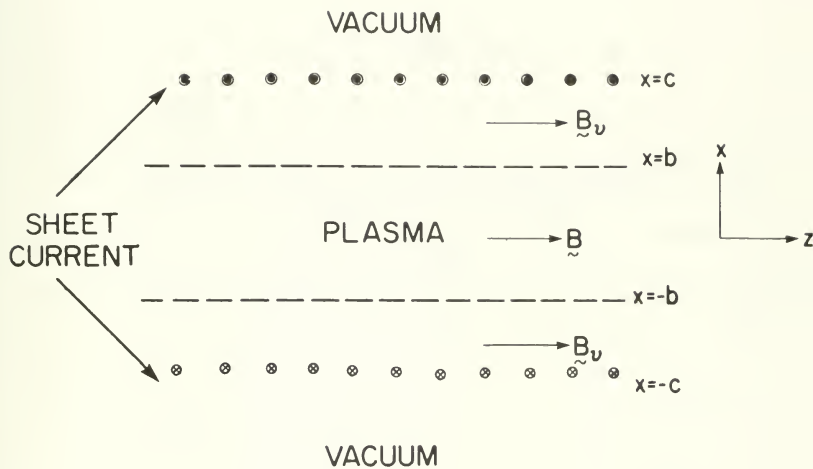


FIG. 1

Plasma Configuration.

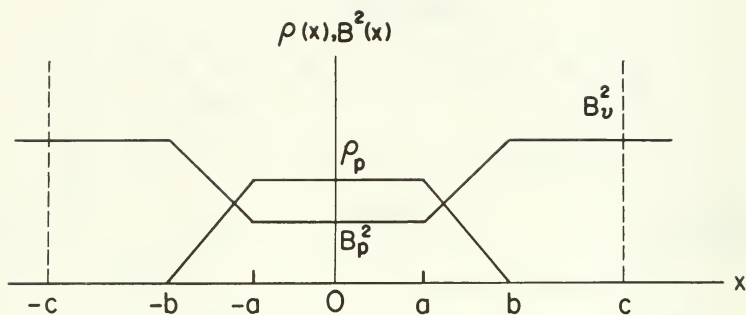


FIG. 2

Piece-wise smooth equilibrium profiles. In the vacuum region $|x| > b$, a sheet current is located at $x = \pm c$.

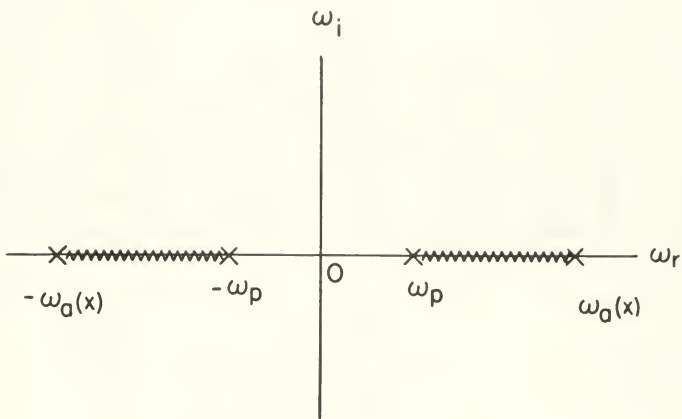


FIG. 3

Branch points ($\times$) and branch cuts (wavy line) of $\psi(x, \omega)$ for the piece-wise smooth profile in Fig. 2.

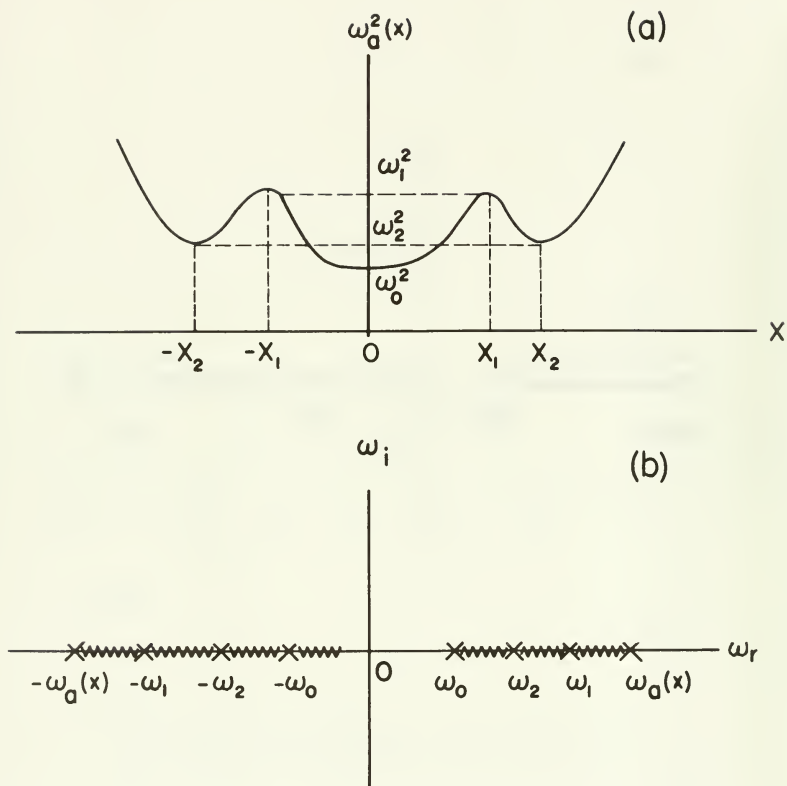


FIG. 4

Branch points (x) and branch cuts (wavy line) of $\psi(x, \omega)$ (Fig. (b)) for the case where ω_a^2 depends on x in the form shown in Fig. (a).

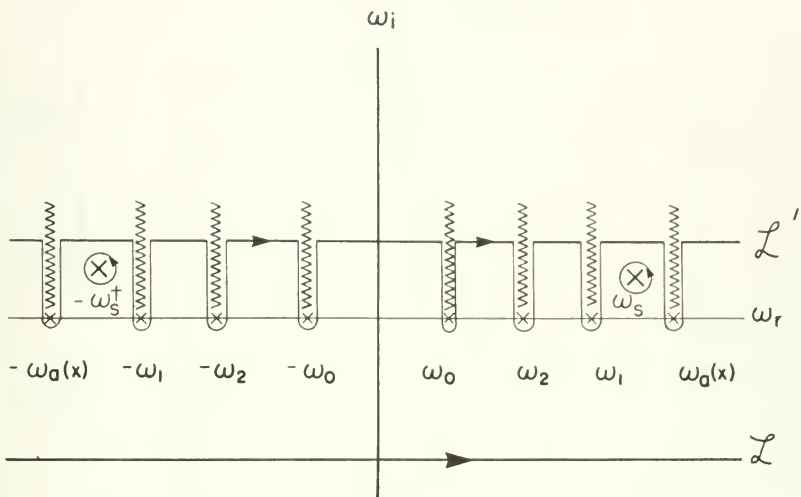


FIG. 5

The contour L along which the inverse Laplace transform is to be accomplished and the contour L' obtained by deforming L around the singularities of $\hat{v}_x(x, \omega)$.

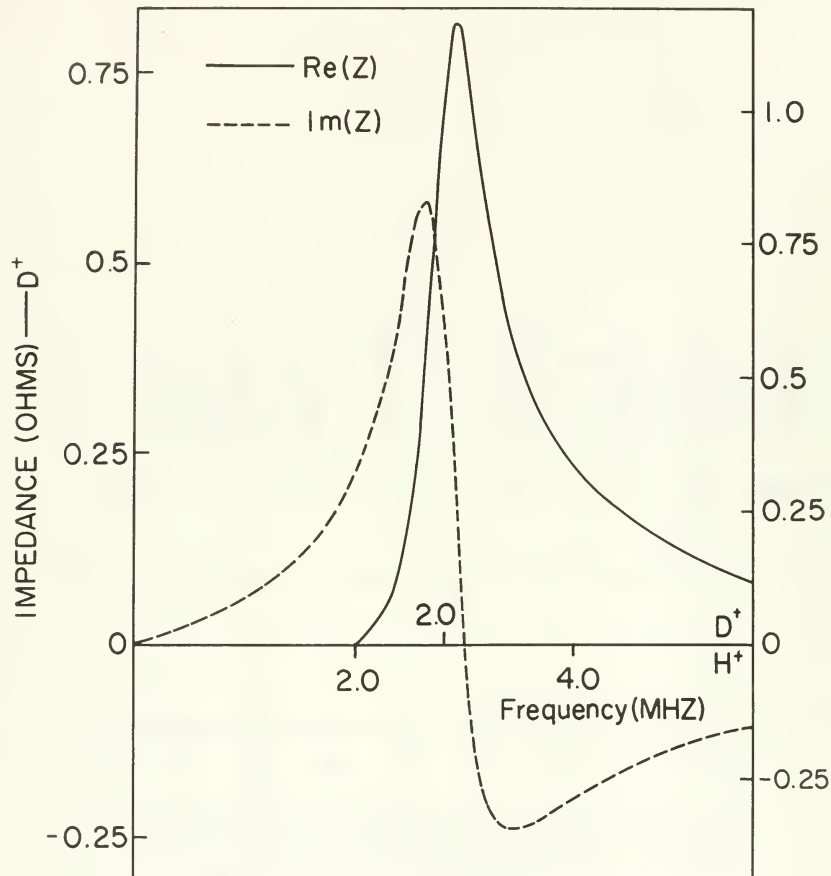


FIG. 6

Coil impedance Z vs. frequency for a hydrogen (H^+) and a deuterium (D^+) plasma. The plasma profile is shown in Fig. 2 with the following parameter values assumed:

$a = 2\text{cm}$, $b = 2.4\text{cm}$, $c = 2.6\text{cm}$, $B_v = 15\text{kg}$, $B_p^2/B_v^2 = 0.5$, $k = 0.5\text{cm}^{-1}$ and $n = 10^{16}\text{p/cm}^3$ where n is the equilibrium particle density.

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